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Holomorphic solutions to pantograph type equations with neutral fixed points

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Abstract

Pantograph type equations have been studied extensively owing to the numerous applications in which these equations arise. These studies focused primarily on the case when the functional argument is linear, and the origin is either a repelling or attracting fixed point. The nonlinear case has been studied by Oberg [Trans. Amer. Math. Soc. 161 (1971) 302–327] and Marshall et al. [J. Math. Anal. Appl. 268 (2002) 157–170], but the focus again was on repelling or attracting fixed points. Oberg (op. cit.), however, did consider briefly the neutral fixed point case and found a connexion with Siegel discs. In this paper we build on Oberg's work and study the neutral fixed point case. We show that, for nonlinear functional arguments with neutral fixed points, pantograph type equations have nonconstant holomorphic solutions only if the functional argument has a Siegel disc centered at the fixed point. We then show that the boundary of the Siegel disc forms a natural boundary for the nonconstant holomorphic solutions.

1. Introduction

The pantograph equation

$$y'(z) + by(z) = \lambda y(\alpha z), \tag{1.1}$$

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where b, λ and α are constants, has been studied by numerous authors (e.g., [7,10–12]). Matrix and second order versions of this equation have also been studied (e.g., [2,24,25]). The enduring interest in this equation is due partially to the number of applications it has found ranging from a current collection system for an electric locomotive [6,18] to cell growth models [9,25]. Though some forays were made in the complex plane [3,10,16], the focus of most of these studies was on solutions on the real line for either the retarded case $0 < \alpha < 1$ or the advanced case $\alpha > 1$.

In this paper we consider a generalization of the pantograph equation

$$y'(z) + by(z) - \lambda y\big(g(z)\big) = 0, \tag{1.2}$$

where b and λ are complex constants with $\lambda \neq 0$, and g is an entire function with a fixed point at $z = z_0$.

Special cases of linear functional differential equations such as (1.2) have been considered by Utz [22], Gross [8], and Oberg [17]. More recently, Rogers [19] and van Brunt [23] studied coupled systems of nonlinear functional differential equations, that include the above equation as a special case. The latter authors concentrated on the local theory for the existence of solutions near a fixed point.

Let g be a function holomorphic at z_0 and let z_0 be a fixed point for g. Recall that *attracting* fixed points are characterized by

$$\big|g'(z_0)\big| < 1,$$

and *repelling* fixed points by

$$\big|g'(z_0)\big| > 1.$$

Neutral (also called indifferent) fixed points are characterized by

$$\big|g'(z_0)\big| = 1.$$

The following is a basic local existence result concerning solutions to equations such as (1.2) when z_0 is an attracting fixed point.

Theorem 1.1. *Let g, p and q be functions holomorphic at z_0 with $g(z_0) = z_0$. Suppose that z_0 is an attracting fixed point for g. Then, for any $y_0 \in \mathbb{C}$ there exists a unique solution y to the equation*

$$y'(z) + p(z)y(z) + q(z)y\big(g(z)\big) = 0, \tag{1.3}$$

that is holomorphic at z_0 and satisfies $y(z_0) = y_0$.

A proof for a special case of this result can be found in [17]. The result has been proved for more general cases by [14] and [23].

The situation is not as clear if z_0 is a repelling fixed point. The local arguments leading to the proof of Theorem 1.1 (in particular an application of the contraction mapping principle) cannot be used in this case. Indeed, for a special case Oberg [17] showed that generically local holomorphic solutions do not exist for the repelling fixed point case. For neutral fixed points, the situation is also complicated, but a careful reading of the proof given by [14, Theorem 2-2, pp. 160–162] shows that the crucial property needed by g is the existence of a positive number δ_1 such that g maps the disc $D(z_0;\delta)$ to $D(z_0;\delta)$ for all $\delta < \delta_1$, i.e., $D_g(z_0;\delta) = \{g(z)\colon z \in D(z_0;\delta)\} \subseteq D(z_0;\delta)$. Now, an attracting fixed point always satisfies this condition for δ_1 sufficiently small, and this case $D_g(z_0;\delta) \subset D(z_0;\delta)$. But the proof is valid if g is such that $D_g(z_0;\delta) \subseteq D(z_0;\delta)$, and this leaves an opening for the neutral case. In summary, we have the following result.

Corollary 1.2. *Let g, p and q be as in Theorem* 1.1. *Suppose that z_0 is a fixed point for g and that there is $\delta_1 > 0$ such that $D_g(z_0;\delta) \subseteq D(z_0;\delta)$ for all $\delta < \delta_1$. Then the conclusions of Theorem* 1.1 *remain valid.*

In the next section we study the linear case, i.e., the pantograph equation (1.1). Simple power series arguments suffice to establish the existence of entire solutions in the attracting and neutral fixed point cases. For the repelling fixed point case we show that no holomorphic solutions are available. Even in this simple framework, the neutral case is intriguing because in certain cases the order of the entire functions depends on the arithmetical properties of the multiplier α, and these properties are much like those used to distinguish Siegel points from Cremer points for nonlinear polynomials. The relationship between the Julia set for the functional argument and the corresponding solutions to Eq. (1.2) is discussed in Section 3. In Section 4 we study Eq. (1.2) when g is a nonlinear polynomial. For the neutral case we show that this equation has nonconstant solutions only if g has a Siegel disc with center at z_0.

Throughout the paper we use $D(a;r)$ to denote the disc centered at $a \in \mathbb{C}$ of radius $r > 0$. We use $\mathcal{H}(\Omega)$ to denote the space of functions holomorphic in the set $\Omega \subseteq \mathbb{C}$.

2. The linear case

We begin our study of Eq. (1.2) for the case when g is of the form $g(z) = \alpha z$, where α is some complex constant. We thus study nontrivial solutions to the pantograph equation (1.1) that are holomorphic at $z = 0$. This equation was studied in depth by Kato and McLeod [12] and later by Iserles [10]. The results discussed here for the nonneutral case are due mainly to these authors.

Suppose that y is a nontrivial solution to Eq. (1.1) that is holomorphic at $z = 0$. Then y can be represented as a power series of the form

$$y(z) = \sum_{n=0}^{\infty} c_n z^n,$$

that has a nonzero radius of convergence. Substituting this power series into Eq. (1.1) and equating coefficients of z^n gives

$$c_n = \frac{c_0}{n!} \prod_{k=1}^{n} (\lambda \alpha^{k-1} - b) \tag{2.1}$$

for $n \geqslant 1$. If $z = 0$ is an attracting fixed point for g then $|\alpha| < 1$; hence,

$$|c_n| \leqslant |c_0| \frac{(|\lambda| + |b|)^n}{n!} \tag{2.2}$$

so that the solution is entire. The order ρ of an entire function is given by

$$\rho = \limsup_{n\to\infty} \frac{n \ln n}{\ln(1/|c_n|)}. \tag{2.3}$$

If $|b| > |\lambda|$ then, for n large,

$$|c_n| \sim \frac{|b|^n}{n!} |c_0|,$$

and the above equation implies that y is of order 1. In fact, if $b \neq 0$ and if $\lambda \neq b/\alpha^j$ for all nonnegative integers j, then the function is also of order 1. If $b = 0$, then

$$y(z) = \sum_{k=0}^{\infty} \alpha^{k(k-1)/2} (\lambda z)^k,$$

and y is of order 0 (cf. Iserles op. cit.); if $\lambda = b/\alpha^j$ then $c_{j+1} = c_{j+2} = \cdots = 0$ so that y is a polynomial of degree j and hence of order 0.

The situation is markedly different if $z = 0$ is a repelling fixed point, i.e., if $|\alpha| > 1$. If y has a nontrivial solution holomorphic at $z = 0$ then $c_0 \neq 0$, and if $\lambda \neq b/\alpha^j$ for any nonnegative integer j then

$$\frac{c_{n+1}}{c_n} = \frac{|\lambda\alpha^n - b|}{n+1} \to \infty$$

as $n \to \infty$; hence, the power series has a zero radius of convergence. In this manner we see that if Eq. (1.1) has a solution holomorphic at $z = 0$ then $\lambda = b/\alpha^j$ for some nonnegative integer j. In this case y must be a polynomial of order j. In contrast with the attracting fixed point case, the only nontrivial holomorphic solutions to (1.1) are polynomials and in this case $\lambda = b/\alpha^j$ for some nonnegative integer j.

We now consider the neutral case when $|\alpha| = 1$. Inequality (2.2) is still valid for $|\alpha| = 1$ and consequently there are nontrivial entire solutions to Eq. (1.1). If $|\lambda| \neq |b|$, then the inequality

$$|c_0| \frac{||\lambda| - |b||^n}{n!} \leqslant |c_n|$$

along with inequality (2.2) show that the entire function is of order 1. If λ is an eigenvalue, i.e., $\lambda = b/\alpha^j$ for some nonnegative integer j, then it is clear from relation (2.1) that the solution is a polynomial. The interesting case is when $|\lambda| = |b|$, but λ is not an eigenvalue. In this case the order of the entire solution depends on α. To illustrate this comment consider the special example

$$y'(z) + y(z) = \alpha y(\alpha z). \tag{2.4}$$

For this equation, the coefficients c_n are given by

$$c_n = \frac{\prod_{k=1}^{n} (\alpha^k - 1)}{n!}.$$

Let $\alpha = e^{2\pi i \beta}$, where β is a real number. If β is rational, we call α a rational rotation; otherwise, α is called an irrational rotation. If α is a rational rotation, then $\alpha^j = 1$ for some j, and consequently the entire solution y must be a polynomial. Suppose that α is an irrational rotation. Evidently y cannot be a polynomial, but the order of y depends on "how close" α approximates a root of unity. Now,

$$\frac{1}{\ln|c_n|} = \ln n! - \sum_{k=1}^{n} \ln|\alpha^k - 1| \geqslant \ln n! - n\ln 2,$$

so that

$$\rho = \limsup_{n\to\infty} \frac{n\ln n}{1/\ln|c_n|} \leqslant \lim_{n\to\infty} \frac{n\ln n}{\ln n! - n\ln 2} = 1;$$

hence, the maximum order for the entire function y is 1. Suppose that α is such that

$$\ln|\alpha^n - 1| = O(\ln n), \tag{2.5}$$

as $n \to \infty$. The above condition corresponds to Siegel's condition for a stable neutral fixed point [21]. In this case there is $L > 0$ such that

$$-\ln|\alpha^n - 1| < L\ln n$$

for n sufficiently large; consequently,

$$\frac{1}{\ln|c_n|} \geqslant \ln n! + nL\ln n$$

for n large. Therefore,

$$\lim_{n\to\infty} \frac{n\ln n}{\ln n! + nL\ln n} = \frac{1}{1+L} \leqslant \limsup_{n\to\infty} \frac{n\ln n}{1/\ln|c_n|}.$$

For such α, the order ρ of y must satisfy

$$\frac{1}{1+L} \leqslant \rho \leqslant 1.$$

Presumably, one can find Cremer type relations for irrational rotations that ensure the function is of order 0. At any rate, the order of the entire function for this example is linked to the arithmetical properties of α.

Finally, it is of interest to note that Liu [13] studied the existence of holomorphic solutions to the Shabat functional equation

$$y'(z) + \alpha^2 y'(\alpha z) + y^2(z) - \alpha^2 y^2(\alpha z) = \text{const}$$

for the neutral case $|\alpha| = 1$. This study brought to the fore Siegel discs. We show in Section 4 that the existence of Siegel discs is also crucial for the existence of holomorphic solutions to pantograph type equations with nonlinear polynomial functional arguments.

3. Analytic continuation and the Julia set

Local holomorphic solutions to functional differential equations such as (1.2) can be extended by iterating the function g. For example, if y is a solution to Eq. (1.2) holomorphic in some neighborhood $N(z_0)$ of a fixed point z_0, then Eq. (1.2) indicates that y is holomorphic in the neighborhood $N_g(z_0) = \{g(z)\colon z \in N(z_0)\}$. If $N(z_0) \subset N_g(z_0)$ then this process provides a nontrivial analytic continuation of the solution. Equation (1.2) can then be recast as

$$y'(g) + by(g) = \lambda y(g_2),$$

where $g_2(z) = g(g(z))$. The same process can then be used to define another neighborhood $N_{g_2}(z_0) = \{g(z)\colon z \in N_g(z_0)\}$ wherein the solution must is holomorphic. Evidently, the extent of the analytic continuation depends crucially on g. Certainly if z_0 is a repelling fixed point for g and $N(z_0)$ is a small disc centered at z_0, then this process will provide a nontrivial analytic continuation. The situation, however, is more extreme when g is nonlinear and z_0 is a repelling fixed point. Indeed, we show below that the solution can be analytically continued throughout the complex plane with at most one exceptional point, where the solution may have a singularity. Briefly, the continuation arises for this case because the point z_0 must be in the Julia set, and any neighborhood of a point in the Julia set can be used to generate a family of neighborhoods that cover the complex plane with at most one exception.

The Julia set $J(g)$ of the entire function g is defined as the closure of the set of repelling periodic points of g. The standard definition of the Julia set is in terms of normal families, but it can be shown that the definitions are equivalent [5,15]. The Fatou set $F(g)$ is defined as $\mathbb{C} \setminus J(g)$. The next theorem gives an important property of Julia sets in connexion with analytic continuation (cf. [4]).

Theorem 3.1. *Let $z \in J(g)$, U be a neighborhood of z, $g_0(U) = \{g(z)\colon z \in U\}$ and $g_n(U) = \{g(z)\colon z \in g_{n-1}(U)\}$ for $n \geqslant 1$. Then the set G defined by*

$$G = \bigcup_{n=1}^{\infty} g_n(U)$$

omits at most one point in $\mathbb{C}$.

Suppose that y is a solution to Eq. (1.2) holomorphic in some neighborhood U containing the fixed point z_0 of g. As discussed earlier, an analytic continuation (possibly trivial) can be made using the relation

$$y'(g_{n-1}) + by(g_{n-1}) = \lambda y(g_n), \tag{3.1}$$

where $g_0(z) = g(z)$ and $g_n(z) = g(g_{n-1}(z))$. If U contains any point in the Julia set, then the above expression implies that the local solution can be continued throughout the complex plane with at most one exception. More formally, we have the following result.

Corollary 3.2. *Let $y \in \mathcal{H}(\Omega)$ be a solution to Eq.* (1.2). *If $J(g) \cap \Omega \neq \emptyset$ then y can be analytically continued to all points in the complex plane with at most one exception.*

If there is $\sigma \in \mathbb{C}$ such that $G = \mathbb{C} \setminus \{\sigma\}$, then σ is called an *exceptional point* for g. The characterization of the exceptional point is particularly simple for polynomials (cf. [4]).

Theorem 3.3. *Let g be a polynomial. Suppose that there is a point $z \in J(g)$ and a neighborhood U of z such that*

$$\bigcup_{n=1}^{\infty} g_n(U) = \mathbb{C} \setminus \{\sigma\}$$

for some $\sigma \in \mathbb{C}$. Then $g(z) = \sigma + \kappa(z - \sigma)^k$ for some $\kappa \in \mathbb{C} \setminus \{0\}$ and some $k \in \mathbb{N}$.

If σ is an exceptional point for a polynomial g and y is a solution to Eq. (1.2) holomorphic at some point $z \in J(g)$, then y can be continued to all points in the complex plane by Eq. (3.1) except σ. The exceptional point may be an isolated singularity, a branch point, or a point at which y is holomorphic. The following simple examples illustrate these possibilities.

Example 1. Let $g(z) = z^2$. Then g has an attracting fixed point at $z_0 = 0$ and a repelling fixed point at $z_1 = 1$. Since z_1 is a repelling fixed point, $z_1 \in J(g)$. If g has an exceptional point it must be z_0. Consider the equation

$$y'(z) + y(z) = y(z^2). \tag{3.2}$$

Certainly, any function $y = \text{const}$ is a solution to this equation. These solutions are entire so that in particular y is holomorphic at z_0 and z_1.

Example 2. Consider the equation

$$y'(z) = -y(z^2). \tag{3.3}$$

Any function of the form $y(z) = c/z$, where c is a constant, is a solution to this equation. For this case the solution is holomorphic at the repelling point but has a simple pole at the exceptional point.

Example 3. Let $g(z) = z^3$. Now, g has an attracting fixed point at $z_0 = 0$ and repelling fixed points at $z_1 = 1$ and $z_2 = -1$. The fixed points z_1 and z_2 are in $J(g)$, and if g has an exceptional point it must be at z_0. Consider the equation

$$y'(z) = -y(z^3). \tag{3.4}$$

This equation has solutions of the form $y(z) = c/\sqrt{z}$, where c is a constant, which are holomorphic at z_1. In this case the exceptional point is a branch point.

For polynomials, Theorems 3.1, 3.3 and Corollary 3.2 indicate that, generically, solutions to Eq. (1.2) that are holomorphic at a point in the Julia set must be entire functions. This leads one to query about the nature of the entire functions thus produced. The next result is valid for any nonlinear entire function g.

Theorem 3.4. *Let g be a nonlinear entire function and suppose that y is an entire solution to Eq.* (1.2). *Then y must be a constant function.*

Proof. For any entire function f let

$$M_f(R) = \sup_{|z|=R} |f(z)|.$$

We prove the theorem for the case when $g(0) = 0$ and note how the proof can be extended. For this case, Polya [20] showed that for any entire functions y and g there is a constant c, $0 < c < 1$, such that

$$M_{y\circ g}(R) \geqslant M_y\left(cM_g\left(\frac{R}{2}\right)\right) \tag{3.5}$$

for all $R > 0$. Suppose that y is a nonconstant solution to Eq. (1.2) and let h be the function defined by

$$h(z) = \begin{cases} \frac{y(z)-y(0)}{z} & \text{if } z \neq 0, \\ y'(0) & \text{if } z = 0. \end{cases}$$

Since y is entire and nonconstant h is an entire function. The above definition of h provides the inequalities

$$M_y \leqslant RM_h(R) + |y_0| \tag{3.6}$$

and

$$RM_h(R) \leqslant M_y(R) + |y_0| \tag{3.7}$$

for all $R > 0$.

The Cauchy integral formula can be used to establish the inequality

$$M_{y'}(R) \leqslant \frac{M_y(R+\delta)}{\delta},$$

where $\delta > 0$, and Eq. (1.2), the maximum modulus theorem and the Polya inequality (3.5) imply

$$M_y(R+\delta)\left(\frac{1}{\delta} + |b|\right) \geqslant |\lambda| M_{y\circ g}(R) \geqslant |\lambda| M_y\left(cM_g\left(\frac{R}{2}\right)\right). \tag{3.8}$$

Inequalities (3.6)–(3.8) thus yield

$$\big((R+\delta)M_h(R+\delta) + |y_0|\big)\left(\frac{1}{\delta} + |b|\right) \geqslant |\lambda|\left(cM_g\left(\frac{R}{2}\right)M_h\left(cM_g\left(\frac{R}{2}\right)\right) - |y_0|\right),$$

and since $M_h(R) \neq 0$ and $M_g(R/2) \neq 0$ for any $R > 0$, the above inequality can be recast as

$$M_h(R+\delta)L(R) \geqslant M_h\left(cM_g\left(\frac{R}{2}\right)\right),$$

where

$$L(R) = \left\{\frac{1}{|\lambda|}\left(R + \delta + \frac{|y_0|}{M_h(R+\delta)}\right)\left(\frac{1}{\delta} + |b|\right) + \frac{|y_0|}{M_h(R+\delta)}\right\}\frac{1}{cM_g(R/2)}.$$

Now, g is nonlinear and hence $R/M_g(R/2) \to 0$ as $R \to \infty$. Consequently there is $\hat{R} > 0$ such that

$$cM_g\left(\frac{R}{2}\right) > R + \delta$$

and

$$L(R) < 1$$

for all $R \geqslant \hat{R}$. For such R we thus have

$$M_h(R+\delta) > M_h\left(cM_g\left(\frac{R}{2}\right)\right),$$

which contradicts the maximum modulus theorem. We thus conclude that y must be constant.

For the case $g(0) \neq 0$, we can use the transformations given by Gross [8]. In fact, the condition $g(0) = 0$ is needed only for the Polya inequality, and we can avoid the use of it by simply noting that since g is nonlinear there are numbers β and $\hat{R}$ such that

$$M_{y \circ g}(R) \geqslant M(\beta R^2)$$

and

$$\beta R^2 > R + \delta$$

for all $R > \hat{R}$. This approach was used to prove a similar result in Marshall et al. [14]. □

4. Nonlinear polynomial arguments

We now consider the case where g is a nonlinear polynomial. For definiteness, we assume that the fixed point of interest is at the origin, noting that the results in this section transfer immediately to fixed points at other locations. Thus, we consider functional arguments of the form

$$g(z) = \alpha z + a_2 z^2 + \cdots + a_n z^n, \tag{4.1}$$

where $n > 1$, $a_n \neq 0$.

Let $z_0 = 0$. If $|\alpha| < 1$, then z_0 is an attracting fixed point. In this case it can be shown that Eq. (1.2) has nontrivial solutions holomorphic at the origin and that these solutions can be continued throughout the connected component of the basin of attraction for $z = 0$. The boundary of the connected component of this basin, which is part of the Julia set for g, forms a natural boundary for the solution (cf. Marshall et al. [14]). The exceptional case is when $\lambda = b$. In this case there are constant solutions.

If $|\alpha| > 1$, then z_0 is repelling and hence an element of the Julia set. Thus, if there exists a nontrivial solution holomorphic at z_0 we have by Corollary 3.2 that this solution can be continued throughout the complex plane with at most one exception. If the solution is entire, then by Theorem 3.4 it must be a constant solution, which is available only if $\lambda = b$. If $\lambda \neq b$ we thus see that there is a nontrivial solution to Eq. (1.2) holomorphic at the origin only if there is an exceptional point. If there is an exceptional point then Theorem 3.3 indicates that there are numbers $\sigma, \kappa \in \mathbb{C} \setminus \{0\}$ such that

$$g(z) = \sigma + \kappa(z - \sigma)^n,$$

and the conditions $g(0) = 0$, $g'(0) = \alpha$ imply that

$$0 = \sigma + (-1)^n \kappa \sigma^n, \qquad \alpha = n\kappa(-1)^{n-1}\sigma^{n-1}$$

so that $\alpha = n$.

We now focus on the neutral case when $|\alpha| = 1$. This case presents different problems because a neighborhood of z_0 may or may not contain elements of the Julia set depending on the arithmetical properties of α. Note that g cannot have an exceptional point in this case since this would require $\alpha = n$, and by hypothesis $n > 1$. We can combine this observation with Corollary 3.2 and Theorems 3.3, 3.4 to glean the following result.

Lemma 4.1. *Let g be as defined by* (4.1), *$\lambda \neq b$, and $|\alpha| = 1$. There exists a nontrivial solution y holomorphic at z_0 only if $z_0 \in F(g)$.*

Evidently, if $\lambda = b$ then the constant solution is available; otherwise, the above theorem indicates that nontrivial holomorphic solutions exist only if z_0 is in the Fatou set. The circumstances under which $z_0 \in \mathbb{C} \setminus J(g)$ are special. Let Ω be the connected component of the Fatou set containing z_0. Briefly, it can be shown [15, p. 116] that the condition $z_0 \in F(g)$ is equivalent to the condition that g is locally linearizable about z_0, i.e., there is a coordinate transformation $z = h(w)$ such that

$$g\big(h(w)\big) = h(\alpha w), \tag{4.2}$$

where h is holomorphic at $w = 0$, $h(0) = 0$ and $h'(0) \neq 0$. Moreover, $h \in \mathcal{H}(D(0;1))$ and h is a one-to-one mapping from $D(0;1)$ to Ω. Given the existence of such a transformation, Eq. (1.2) can be recast in the form

$$Y'(w) + bh'(w)Y(w) - \lambda h'(w)Y(\alpha w) = 0, \tag{4.3}$$

where $Y(w) = y(h(w))$ and $'$ denotes d/dw.

Lemma 4.2. *For any $Y_0 \in \mathbb{C}$, there exists a unique solution $Y \in \mathcal{H}(D(0;1))$ to Eq.* (4.3) *that satisfies $Y(z_0) = Y_0$.*

Proof. Since $h \in \mathcal{H}(D(0;1))$ and $\{\alpha z\colon z \in D(0;\delta)\} = D(0;\delta)$ for any $\delta > 0$, the conditions of Corollary 1.2 are satisfied. Hence, for any $Y_0 \in \mathbb{C}$ there is $\epsilon > 0$ and a function $Y \in \mathcal{H}(D(0;\epsilon))$ such that Y is a solution to Eq. (4.3) and satisfies $Y(0) = Y_0$. Moreover, Y is unique.

Corollary 1.2 guarantees a unique solution to the initial-value problem, but the result is local in character and it remains to show that the holomorphic solution can be analytically continued in to the disc $D(0;1)$. If Eq. (4.3) were an ordinary differential equation we could appeal to well-known results concerning the analytic continuation of solutions to discs of diameter at least that of the radius of convergence of h (cf. [1]). Nonetheless, we can mimic the proof of this result for the neutral case and employ the method of majorants to deduce that $Y \in \mathcal{H}(D(0;1))$. Specifically, we have that Y can be represented in the form

$$Y(w) = \sum_{n=0}^{\infty} C_n w^n$$

for $w \in D(0;\epsilon)$, and that h' can be represented in the form

$$h'(w) = \sum_{n=0}^{\infty} k_n w^n$$

for $w \in D(0;1)$. Substituting these power series into Eq. (4.3) gives

$$C_{n+1} = \frac{1}{n+1} \sum_{j=0}^{n} k_{n-j} C_j (\lambda \alpha^j - b). \tag{4.4}$$

For any r, $0 < r < 1$ there is a number Λ_r such that

$$|k_n| \leqslant \Lambda_r r^{-n}$$

for all n. Let $w \in D(0;r)$, $q = |\lambda| + |b|$,

$$K_n = q \Lambda_r r^{-n},$$

and define the function H as

$$H(w) = \sum_{n=0}^{\infty} K_n w^n = q \Lambda_r \sum_{n=0}^{\infty} \left(\frac{w}{r}\right)^n = \frac{q \Lambda_r}{1 - w/r}.$$

By construction, the differential equation

$$P'(w) + H(w)P(w) = 0 \tag{4.5}$$

majorizes Eq. (4.3) under the condition

$$P(0) = |C_0|, \tag{4.6}$$

so that if P is represented by the power series

$$P(w) = \sum_{n=0}^{\infty} b_n w^n,$$

then $b_n > |c_n|$ for all n. Now, Eq. (4.5) with the initial condition (4.6) has the unique holomorphic solution

$$P(w) = \left(1 - \frac{w}{r}\right)^{-rq\Lambda_r} |C_0|.$$

Since $P \in \mathcal{H}(D(0;r))$ and r can be arbitrarily close to 1, then we conclude that $Y \in \mathcal{H}(D(0;1))$. $\square$

We can thus strengthen Lemma 4.1 as follows.

Theorem 4.3. *Let g be as defined by* (4.1), $\lambda \neq b$ *and* $|\alpha| = 1$. *Then there exists a nontrivial solution y to Eq.* (1.2) *holomorphic at z_0 if and only if g is locally linearizable about z_0.*

In addition, Lemma 4.2 and Theorem 3.4 highlight the rôle of the Siegel disc Ω and its boundary $\partial\Omega$.

Corollary 4.4. *Under the conditions of Theorem* 4.3, *suppose that y is a nontrivial solution to Eq.* (1.2). *Then $y \in \mathcal{H}(\Omega)$, and $\partial\Omega$ is a natural boundary for y.*

Proof. The function h is a conformal isomorphism from $D(0; 1)$ to Ω, and, by Lemma 4.2, $Y \in \mathcal{H}(D(0; 1))$; hence, $y \in \mathcal{H}(\Omega)$. Now $\lambda \neq b$ so that y is not a constant function; consequently, Theorem 3.4 shows that y cannot be holomorphic at any point of the Julia set. By construction $\partial\Omega \subseteq J(g)$, and we thus conclude that y is singular at every point of $\partial\Omega$. □

The conditions under which g is locally linearizable near z_0 have been studied extensively. The reader is directed to Milnor [15] or Devaney [4,5] for an overview. Briefly, if α is a rational rotation, then results such as the Leau–Fatou theorem preclude the possibility of holomorphic solutions. Essentially, there are repelling and attracting periodic points arbitrarily close to z_0 in this case and hence $z_0 \in J(g)$. The example given by Oberg [17] ($g(z) = z + z^2$) is of this type, though he argues directly through power series expansions that the differential equation cannot have a holomorphic solution.

Theorem 4.3 indicates that Eq. (1.2) has nontrivial holomorphic solutions only if α is an irrational rotation. Not every such number, however, yields a local linearization. The irrational rotations can be partitioned into those that lead to local linearization and those that do not. The former case corresponds to Siegel points; the latter case corresponds to Cremer points. There are results such as the theorem of Bryuno [15, p. 122] that help identify Siegel points. If g is quadratic, then the theorem of Yoccoz ([15], loc. cit.) can be used to identify Cremer points. The generic situation is that irrational rotations are Cremer points, so that in this context it is a "rare" occurrence when the functional differential equation (1.2) has a solution holomorphic at the origin.

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